LEARN: MEASUREMENT - Advanced

Despite its centenary life, QP is still a theory that presents mysteries and open fundamental questions, even for people working with it and exploiting it everyday. A large part of the still-existing doubts is connected with the crux of this theory, the notorious "quantum measurement problem".

While there aren't particular issues with the first and the second postulate, since they deal with the mathematical structure of the theory for which is easy to find a common accepted ground, with the third one all chickens come home to roost. The third postulate is indeed the one about measurement and it is made of two parts: the first one, on which everyone agrees, is contained in all the existing variations of the quantum measurement process and that's why it is called *minimal interpretation*; the second one is the more delicate part, because it is related to the different interpretations of the theory. To get an idea, have a look at the page "Interpretations of Quantum Mechanics" on Wikipedia (https://en.wikipedia.org/wiki/Interpretations_of_quantum_mechanics), where you can find a good recap and a useful table with comparisons between the different interpretations. Among the most famous ones, we mention the Copenhagen and the many-worlds. But the point is that, at least for now, there is no way to experimentally prove which interpretation of QP is the correct one: that's why you are free to choose your favourite one. The reason why we can be quite anarchist about the second part of the third postulate, is that the existence of several interpretations does not undermine at all the solidity of the theory itself. Therefore, we can initially avoid worrying too much about them, and focus instead on the minimal interpretation.

Let us consider a quantity that characterises the physical system, specified by some values. In order to get information on such values, i.e., in order to acquire knowledge on the system, we need to perform some measurements. The quantities that we can measure are called "observables": in QP there is indeed a tight connection between the observation and the measurement, to the point that we can consider *observing* and *measuring* as synonyms. For example, the energy, the position, and the momentum of a system are some of its observables. Performing a measurement on a system means that there is an experimental apparatus that associates a number, m_ℓ, the result, to every measurement. In QP the measurement is indeed an active operation: mathematically, in order to get a result, we need to act on the state of the system with an operator that, loosely speaking, is able to "extract a number". A particularly simple operator that can be associated to each result is a *projector,* $|\ell\rangle\langle\ell|$. Projectors are operators such that

$$\sum_\ell |\ell\rangle\langle\ell| = \hat{\mathbb{1}} \text{ with } \langle\ell|\ell'\rangle = \delta_{\ell\ell'},$$

implying that $\{|\ell\rangle\}_{\mathcal{H}}$ is an orthonormal basis for the system's Hilbert space $\mathcal{H}$. Given an observable M and a set of its results $\{m_\ell\}$, a *measurement* is an application that associates to each result a different projector, namely

$$\{m_\ell\} \to |\ell\rangle\langle\ell| \text{ with } \sum_\ell |\ell\rangle\langle\ell| = \hat{\mathbb{1}}\,.$$

The **third postulate of quantum mechanics** (minimal interpretation) states that the probability of obtaining the result m_ℓ when performing a measurement on the observable M on a system in state $|\Psi\rangle$ is

$$p^M_{|\Psi\rangle}(m_\ell) = \langle\Psi|\ell\rangle\langle\ell|\Psi\rangle = |\langle\ell|\Psi\rangle|^2.$$

The property $\sum_\ell |\ell\rangle\langle\ell| = \hat{\mathbb{1}}$ and the normalisation of the states, $\langle\Psi|\Psi\rangle = 1$, guarantee that the sum of the probabilities related to the whole set $\{m_\ell\}$ gives 1, i.e., $\sum_\ell p^M_{|\Psi\rangle}(m_\ell) = 1$, as it must be the case for a probability distribution.

Given that $\{|\ell\rangle\}_{\mathcal{H}}$ is an orthonormal basis of the Hilbert space, every state $|\Psi\rangle$ in $\mathcal{H}$ can be written as a linear combination of such basis states,

$$|\Psi\rangle = \sum_\ell c_\ell |\ell\rangle \text{ with } \sum_\ell |c_\ell|^2 = 1.$$

Inserting this expression into the previous one, we get

$$p^{M}_{|\Psi\rangle}(m_\ell) = |\langle \ell | \Psi \rangle|^2 = \left(\sum_{\ell'} c^*_{\ell'} \langle \ell' | \right) | \ell \rangle \langle \ell | \left(\sum_{\ell''} c_{\ell''} | \ell'' \rangle \right) = \sum_{\ell'} \sum_{\ell''} c^*_{\ell'} c_{\ell''} \delta_{\ell' \ell} \delta_{\ell \ell''} = |c_\ell|^2,$$

which is the so-called **Born's rule**. Keeping in mind such rule and the decomposition of the state $|\Psi\rangle$, the third postulate can be expressed as follows: "when performing a measurement, the probability of observing the outcome corresponding to state $|\ell\rangle$ is given by the square modulus of its coefficient $|c_\ell|^2$".

Pay attention! This doesn't mean that such probability represents the probability for the system *to be* in the state $|\ell\rangle$. As it follows from the first postulate, the system is in the state $|\Psi\rangle$, which is perfectly defined even if rewritten as a sum of several kets: it's the superposition principle. The result m_ℓ related to the state $|\ell\rangle$ represents only one of the possible measurement outcomes, since it is associated to one of the kets composing the state $|\Psi\rangle$. Obtaining a given result from the measurement doesn't imply, however, that the system *was* in the corresponding state before the measurement. The probability stems from the superposition, and QP only tells us which is the probability to get a certain result when we perform a measurement. As odd as it may seem, this is all that the theory allows us to know. Nevertheless, it is relevant to notice again that it's only at this stage that the probabilistic part of the theory enters the scene: before observing the system, everything is deterministic.

Once the measurement is performed and the result m_ℓ is obtained, you may naturally wonder what happens to the system that we observed. The answer is *it depends*. From this point onwards, we enter the territory of the different interpretations: as a matter of fact, QP foresees very different fates for the observed systems according to the chosen variation.

A phenomenon that you might have heard in connection to quantum measurement is the so-called "wave-function collapse", or, in more modern terms, **quantum state reduction**. It's included in the greater part of the interpretations and it's due to the action of the measurement:

$$|\Psi\rangle = \sum_\ell c_\ell |\ell\rangle \xrightarrow[M,\, m_\ell]{} |\ell\rangle \text{ with probability } p^{M}_{|\Psi\rangle}(m_\ell) = |c_\ell|^2,$$

which, in words, means that after the measurement the system will be in the state $|\ell\rangle$, if the measurement result is m_ℓ. It is as if the measurement is so invasive as to destroy the initial superposition. Of course, this can eventually happen if and only if the system keeps existing, since often physical measurements destroy the system itself, as in the case of a photon revealed by a detector.

One of the most famous interpretations, the Copenhagen one, spread the belief that we do not see quantum superpositions because QP holds true on smaller scales (of energy and time) with respect to those typical of the experiments. Today we know that this is not exactly the case: QP does not hold true on small scales only, but independently on the scale. When the number of particles that compose a quantum system becomes very large, the system generally displays physical behaviours that we can explain with Classical Mechanics. However, there are nowadays several examples of macroscopic physical systems that display, instead, an intrinsically quantum character.

We know that a single measurement on a quantum system does not tell us much about it: the system is generally not described by a single state $|\ell\rangle$, but by a linear combination of those basis vectors. It's the vectorial structure itself that, in general, forbids us to obtain a deterministic answer when we interrogate the system through the measurement. In order to obtain the complete probability distribution, we must repeat the same experiment, performed in exactly the same way, many times. Nevertheless, notice that cases in which the measurement outcome is certain do actually exist: whenever the state of the system is one of the states $|\ell\rangle$ corresponding to one of the projectors

$|\ell\rangle\langle\ell|$ of the measurement we are performing. In this situation, we obtain outcome m_ℓ with certainty, since $p^M_{|\Psi\rangle}(m_\ell) = |c_\ell|^2 = 1$.

To every observable M it is possible to associate a measurement operator $\hat{M}$ defined in terms of the sum of the projectors $|\ell\rangle\langle\ell|$, each related to the respective outcome m_ℓ, as

$$\hat{M} = \sum_\ell m_\ell |\ell\rangle\langle\ell| .$$

Since the results m_ℓ of the measurements are always real numbers, the operator $\hat{M}$ is Hermitian, i.e., $\hat{M} = \hat{M}^\dagger$. The decomposition above is the one given by the spectral theorem. The operator $\hat{M}$ gives a certain answer when the system is in one of its eigenstates $\{|\ell\rangle\}_{\mathscr{H}}$; so if this is the case, we can get only one possible result performing the measurement, and it will always be the same every time we repeat the measurement. Therefore, in a sense, the answer that the system gives to the question we ask when we perform a measurement may be probabilistic or not. When we do not know anything about the physical system under analysis, it's really difficult to find "the right question", namely a question which does not give a probabilistic answer; when instead we control the preparation of the system, for example, if we make an experiment to generate a given quantum state with high precision, the scenario changes completely. This is in fact what generally the experimental physicists do when they design their experiments, helped by the theoretical physicists who create representative models of the different physical systems. Concluding, if you find and ask the "right question", the answer that the system gives is deterministic. As Born replied to Einstein's objection "God does not play dice with the universe": "Not only God plays dice, but they are also fixed".

Repeating the measurements many times, we can infer the complete probability distribution $\{m_\ell\} \to p^M_{|\Psi\rangle}(m_\ell)$ through which it is possible to calculate the mean value of observable $\hat{M}$. Indeed, exploiting the ket-bra notation, we define the **expectation value** of $\hat{M}$, as

$$\overline{M} = \langle\hat{M}\rangle = \langle\Psi|\hat{M}|\Psi\rangle = \langle\Psi|(\sum_\ell m_\ell |\ell\rangle\langle\ell|)|\Psi\rangle = \sum_\ell m_\ell p^M_{|\Psi\rangle}(m_\ell).$$

This expression is akin to the definition of mean values for classical stochastic processes. Moreover, although here we used discrete variables, generalising to the continuous case is simple. It is only a formal issue, not related to the definition of measurement itself. Whether the energy spectrum is discrete or continuous depends solely upon the system's features.

Let us conclude by noticing one of the peculiarities of QP, constituting the essence of the concept of measurement, namely the active role of the observer in the process. In order to perform a measurement the system and the apparatus need to interact, so that the latter can acquire information on the former. In the classical case, an idealised perfect measurement reveals the physical properties of the system without altering its state. In QP, this is fundamentally impossible in general. An idealised perfect quantum measurement generally changes the state into an eigenstate of the observable $\hat{M}$. In this sense, at variance with classical systems, it is not possible to identify the measurement outcome with the state of the system prior to the measurement.