LEARN: QUANTUM STATE - Advanced

A physical system can be any part of the Universe that we want to study: a planet, a car, an atom, an electron. What is not included in the chosen physical system constitutes its environment. The environment and the physical system generally *interact* with each other. Nevertheless, it can happen that such interactions are not significant enough to have a noticeable effect on the physical system that we aim to analyse and therefore they can be ignored. In these cases we say that the system is "isolated": the only parts that interact are the inner parts of which the system is composed, whereas everything else can be omitted.

Before deepening into quantum theory, let us notice that the proper language of physics is mathematics. Perhaps it may seem surprising but, of all branches of theoretical physics, quantum physics is the one using the simplest mathematical formulation: all we need is linear algebra. Let us recall very briefly some of the tools that we are going to use. We'll start from the Hilbert space, $\mathscr{H}$, which is a complex vector space where an inner product is defined, i.e., a product between the elements of the space. Being $\mathscr{H}$ a vector space, its elements are vectors and, to identify them, we use the bra and ket notation introduced by P. A. M. Dirac. The symbol of the *ket* is $|\cdot\rangle$, so that considering its dual $\langle\cdot|$, named *bra,* the inner scalar product of the Hilbert space is identified via $\langle\cdot|\cdot\rangle$, known as *braket.* The elements of $\mathscr{H}$, the kets, are characterised by the following properties: the sum (adding two kets together gives another ket) and the multiplication by a scalar (the product between a complex number and a ket is another ket); thirdly, the definition of the inner product as a scalar product, i.e. the operation $\langle\cdot|\cdot\rangle$ is an application which associates 2 elements of $\mathscr{H}$ with a complex number. While the first two properties are shared by all vector spaces, the third one characterises $\mathscr{H}$ introducing the concept of norm, i.e., of distance, or difference between two objects. From the last property we can define orthonormality: two vectors $|v\rangle, |v'\rangle$ are orthonormal if $\langle v|v'\rangle = \delta_{vv'}$, with $\delta_{vv'} = 1$ if $v = v'$, 0 otherwise. In order to build the Hilbert space, it's always possible to choose an orthonormal basis $\{|v\rangle\}_{\mathscr{H}}$ in terms of which every other vector of $\mathscr{H}$ can be obtained as a linear combination. The number of vectors included in the basis, or, in other words, the vectors necessary to span the Hilbert space, defines the dimension of $\mathscr{H}$. Lastly, let us notice that the peculiarity of the Hilbert space essential for quantum physics is its property of being *separable*. This means that its dimension is numerable, because it has the same cardinality of the basis: in plain words, we can count it.

The **first postulate of quantum mechanics** defines the states of any isolated physical system: given one such system, we associate to it a Hilbert space $\mathscr{H}$. Each physical state of the system is described by a normalised vector of $\mathscr{H}$, that is, represented as $|\Psi\rangle$, such that $\langle\Psi|\Psi\rangle = 1$. The opposite is also true: each normalised element of $\mathscr{H}$ represents a possible state of a physical system. The normalised vectors are sometimes referred to as *physical states*, while unnormalised vectors are instead exclusively mathematical objects.

We notice that Hilbert spaces associated to different properties of the same physical system can have different dimensions. For example, if we consider an electron in a given potential, the Hilbert space associated to its spin has dimension equal to 2; the Hilbert space associated to its energy has instead infinite dimension.

Describing a physical state by a ket is something fixed by the postulate: it doesn't derive from any theorem and cannot be really demonstrated experimentally, in the sense that what we get from the experiments certainly agrees with this idea, but alternative descriptions may also be possible. In Classical Mechanics, a physical state is a point in the phase-space, e.g., representing a car that at the instant t has a speed of 50 km/h and it is in a certain position on the street; in thermodynamics it could be a point in a pressure-volume diagram, representing a particular condition of the macroscopic parameters pressure and volume in which the system is at a certain time. One could think that it's not a big issue to use a ket or a point for describing the state of a system; as we will see, this apparently innocuous choice is instead a radical one, and it has tremendous consequences which will actually allow us to formally represent the most striking features of quantum physics!

Since the states of physical systems are identified by vectors, there is indeed a huge innovation: we can sum them! In a sense, quantum physics gives us the possibility to operate with the states. It is possible *to do something* with them. This was not the case of the points embodying the classical states: we can sum their coordinates, but not the points themselves. In other words, we cannot *do* anything with classical states, we can only *represent* them.