LEARN: SUPERPOSITION - Advanced

Superposition is one of those bizarre properties of quantum systems that challenge our logic and contradict our everyday experience of the world. From a mathematical perspective, it stems naturally from the first postulate of Quantum Mechanics, not by chance often named *superposition principle*, and represents a radical departure from what we are used to in Classical Mechanics.

As a matter of fact, one of the mathematical properties of vectors is their addition: summing two vectors yields another vector and, also, any vector can be represented as a sum of vectors. But what does this imply for a quantum system A? Given an orthonormal basis $\{|\alpha\rangle\}_{\mathscr{H}}$ of its Hilbert space $\mathscr{H}$, each physical state $|\Psi\rangle$ of the system can be written as a linear combination of the elements of the basis, namely as the sum

$$|\Psi\rangle = \sum_{\alpha} c_\alpha |\alpha\rangle,$$

where c_α are complex numbers such that $\sum_{\alpha} |c_\alpha|^2 = 1$, as imposed by the normalisation of $|\Psi\rangle$, i.e., $\langle\Psi|\Psi\rangle = 1$. The number of vectors in the basis $\{|\alpha\rangle\}_{\mathscr{H}}$ depends on the dimension of the Hilbert space $\mathscr{H}$ associated to the system A.

In order to simplify the notation a bit, we can consider a Hilbert space with dimension equal to 2, i.e., with two basis vectors that we name here $|0\rangle$ and $|1\rangle$. Since they are normalised vectors, with $\langle 0|0\rangle = 1$ and $\langle 1|1\rangle = 1$, they represent two possible physical states of the system. But there is more. Their sum is another vector, meaning that, e.g., the element of $\mathscr{H}$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \tag{1}$$

is well-defined and therefore describes a perfectly legitimate state of system A. The coefficient $\frac{1}{\sqrt{2}}$ is the normalisation coefficient, imposed by $\langle\Psi|\Psi\rangle = 1$. This is the superposition principle in a nutshell.

You might have heard that expression (1) indicates "the possibility of finding the system in the state $|1\rangle$ **OR** $|0\rangle$ with a certain probability". Pay attention: this sentence may be misleading! In the absence of measurements, Quantum Mechanics is deterministic, so there is no randomness in the state $|\Psi\rangle$. The state of the system *is* a superposition of $|1\rangle$ **AND** $|0\rangle$ simultaneously, as physically legitimate as the state $|\Psi\rangle = |0\rangle$. Nonetheless, equation (1) encodes the probability of observing the system in states $|0\rangle$ or $|1\rangle$ when the corresponding measurement is performed (see Measurement entry in Quest). Note, however, that whether the outcome is probabilistic or not depends on the type of measurement performed on the system. In general, the very nature of the measurement process (and the consequent collapse of the wave-function) is still an open problem and, in fact, it is considered to be the Holy Grail of Quantum Physics.

To dig more into what a quantum superposition is from a physical perspective, let's consider the spin of an electron. The spin is a quantum magnetic property that certain particles possess. In the case of electrons, which are spin-1/2 particles, it can only take two values ($+\hbar/2$ or $-\hbar/2$) along any direction of the physical space, either x, y or z. This means that the Hilbert space associated with the electron's spin has dimension 2, its bases have two elements, and states of the form (1) are allowed. If we focus on a specific direction, say z, we can indicate with $|0\rangle$ and $|1\rangle$ the states in which the spin component is pointing in the $+z$ or $-z$ directions, respectively, often referred to as "spin up" and "spin down" states. Suppose that we want to consider some other direction, say x, represented by the basis $\{|+\rangle, |-\rangle\}_{\mathscr{H}}$, with the ket $|+\rangle$ identifying the state in which the spin component is pointing in the $+x$ direction, sometimes called "spin right" state, and the ket $|-\rangle$ the state in which the spin component is pointing in the opposite one, sometimes called "spin left" state. The following relations are true

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \qquad\qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Therefore, if we write the state (1) in the x basis, the superposition "disappears", as the state reads simply $|\Psi\rangle = |+\rangle$. This points towards a concept which is at the core of quantum superposition, namely complementarity: two components of the spin, say along the z and x directions, cannot be simultaneously well defined. It may sound strange, but quantum physics prevents it (see the Quest entry Heisenberg principle). Still, a superposition of two states for which a spin component is well-defined but different, as in the case of states $|0\rangle$ and $|1\rangle$, can be equal to a state for which another spin component is well-defined, like $|+\rangle$. Weird, isn't it? This is the essence of the superposition principle.